Statistical-Mechanical Modeling of Satellite Multibeam Coverage on Earth's Surface

Richard Tjörnhammar

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1 Introduction

The increasing number of satellites in low Earth orbit (LEO) and medium Earth orbit (MEO), such as Starlink and OneWeb constellations, poses the challenge of quantifying the number of beams irradiating a given point on Earth's surface. This problem is relevant for:

- Assessing co-frequency interference between satellites.
- Planning spectrum reuse for multibeam satellites.
- Understanding global coverage patterns.

The goal is to develop a statistical-mechanical framework to predict, for a given point on Earth:

- The number of visible satellites.
- The number of co-frequency beams irradiating that point.

Inputs to the model include:

- Total number of satellites N.
- Probability distributions of satellite latitude $\rho_\phi(\phi)$.
- Probability distributions of satellite altitude $\rho_h(h)$.
- Beam coverage patterns (maximum off-nadir angle $\theta_{\max}$) and frequencies.

2 Geometry of Satellite-Beam Coverage

Consider a ground point at latitude ϕ_g and longitude λ_g, and a satellite at geocentric coordinates (r_i, ϕ_i, λ_i), with $r_i = R_\oplus + h_i$. The satellite's visibility and beam coverage are defined using:

- γ_i – central angle between satellite subpoint and ground point.
- ϵ_i – elevation angle from ground point to satellite.
- θ_i – off-nadir angle from satellite to ground point.

The relationships are:

$$\cos\gamma_i = \sin\phi_g \sin\phi_i + \cos\phi_g \cos\phi_i \cos(\lambda_g - \lambda_i) \tag{1}$$

$$\epsilon_i = \arcsin\left(\frac{r_i \sin\gamma_i}{\sqrt{R_\oplus^2 + r_i^2 - 2R_\oplus r_i \cos\gamma_i}}\right) \tag{2}$$

$$\theta_i = \arccos\left(\hat{\mathbf{r}}_{\text{sat}} \cdot \hat{\mathbf{r}}_{\text{ground}}\right) \tag{3}$$

2.1 Satellite and Ground Point Geometry

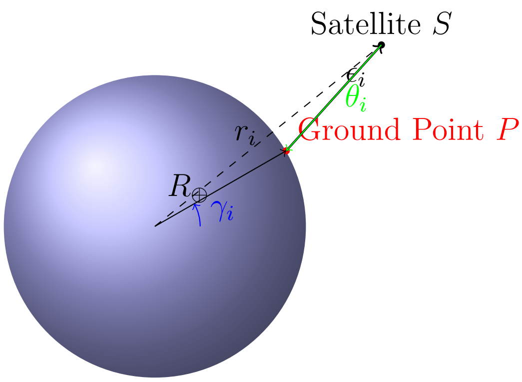

Figure 1: Geometry of satellite, its beams, and a ground point.

3 Multibeam Footprint and Frequency Bands

For multibeam satellites, each satellite has n_b beams arranged in a hexagonal or circular pattern around the sub-satellite point. Beams are assigned frequencies in specific bands (Ku, Ka, E-band), and overlapping footprints on Earth can cause co-frequency interference.

3.1 Beam Footprint Schematic

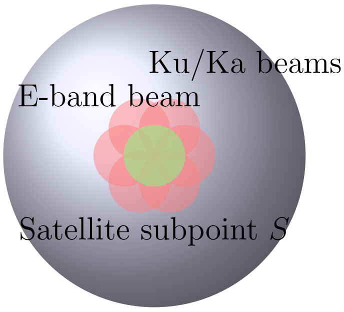

Figure 2: Schematic of multibeam satellite footprint. Colors indicate different frequency bands. Red: Ku/Ka beams; Green: E-band. Overlaps produce co-frequency interference.

4 Statistical-Mechanical Approximation

Assuming independent distributions in latitude $\rho_\phi(\phi)$ and altitude $\rho_h(h)$, the expected number of visible satellites is:

$$N_{\text{vis}}(\phi_g, \lambda_g) = N \int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\phi \int_0^{\infty} dh\; \rho_\phi(\phi)\, \rho_h(h)\, \Theta(\epsilon(\phi, \lambda, h) > 0) \tag{4}$$

Similarly, expected number of beams and co-frequency beams:

$$N_{\text{beams}}(\phi_g, \lambda_g) = \sum_{i=1}^{N} \sum_{b=1}^{n_b} \Theta(\theta_{i,b} \leq \theta_{\max}) \tag{5}$$

$$N_{\text{cofreq}}(\phi_g, \lambda_g; f) = \sum_{i=1}^{N} \sum_{b=1}^{n_b} \Theta(\theta_{i,b} \leq \theta_{\max})\, \delta(f_{i,b} - f) \tag{6}$$

5 Numerical Implementation

The Python implementation includes:

1. **TLE propagation:** Using SGP4 for satellite positions.
2. **Coordinate conversion:** ECI/TEME to ECEF to geodetic coordinates.
3. **Beam placement:** Hexagonal/circular beams per satellite.

4. **Visibility computation:** Off-nadir angles and gain threshold.

5. **Frequency assignment:** Preferred bands (Ku, Ka, E-band) and co-frequency counting.

6. **Chunked GPU processing:** To handle large constellations efficiently.

5.1 Code Highlights

- `assign_beam_frequencies()` – assigns frequency per beam and preferred band marking.
- `beam_gain_offaxis_numpy/cupy()` – computes beam gain for visibility.
- `aggregate_multibeam_visibility_gpu()` – calculates total, preferred-band, non-preferred, and weighted beam counts.

6 Coverage Heatmaps (Optional)

Python-generated global maps of beam counts and co-frequency interference are shown here:

```
\includegraphics[width=\textwidth]{beam_coverage_map.png}
\includegraphics[width=\textwidth]{cofreq_heatmap.png}
```

7 Conclusion

This report presents a statistical-mechanical model for satellite multibeam coverage:

- Satellite distributions and beam geometry are used to compute expected coverage.
- Frequency bands allow computation of co-frequency interference and preferred-band visibility.
- Implementation supports very large constellations ($\sim 35,000$ satellites) with optional GPU acceleration.
- Figures illustrate both single-beam geometry and multibeam footprints with frequency coloring.