Spacecraft low-thrust maneuver detection based on control-distance metric and thrust coefficients

I. Problem statement

For the non-cooperative spacecraft maneuver detection problem using the control-distance metric, Holzinger et al. [1] proposed a control-distance metric to detect and characterize unknown orbital maneuvers for given uncertain boundary states. The probability distribution of the control-distance metric is obtained by using the linear covariance method, which is derived based on the Gaussian distribution assumption. For the non-Gaussian case, Jaunzemis et al. [2] obtained the probability distribution of the control-distance metric by using the Gaussian mixture model (GMM). For each component in the GMM, the probability distribution of the control-distance metric is still gained by using the linear covariance method. For the case with long propagation time and high nonlinearity, a large number of components in the GMM method is needed to guarantee the computation accuracy. In addition, the control-distance metric under given boundary states is obtained by using the indirect method, which may encounter the challenge of guessing the initial co-states.

In this work, we present a spacecraft maneuver detection method based on the control-distance metric and thrust series coefficients. The initial and terminal boundary states are described by using the *mean orbital elements* (e.g., the mean generalized equinoctial orbital elements [3]) to alleviate the nonlinearity of the orbital dynamics. The unknown thrust acceleration profile is expressed as a truncated Fourier series in terms of eccentric anomaly. The control-distance metric under given boundary conditions is obtained by optimizing the thrust series coefficients, and the analytical solution is derived by using the Lagrange multiplier method. In addition, the GMM method is used to capture the non-Gaussian distribution. For each component in the GMM, the *second-order state transition tensor (STT)* is adopted to compute the probability distribution of the control-distance metric.

II. Analytical propagation of mean orbital elements

This section presents the analytical propagation expressions of mean orbital elements under the effects of perturbations (J_2 perturbation is mainly considered) and external thrust acceleration. The results are extracted from Ref. [4].

For the J_2-perturbed two-body problem, the orbital motion under general continuous thrust can be described using the Gauss variation equations (GVEs) [5],

$$\begin{cases} \dfrac{\mathrm{d}a}{\mathrm{d}t} = \dfrac{2\sqrt{a^3}}{\eta\sqrt{\mu}}(e\sin\varphi F_R + \rho F_S) \\ \dfrac{\mathrm{d}e}{\mathrm{d}t} = \dfrac{\eta\sqrt{a}}{\sqrt{\mu}}\left[\sin\varphi F_R + \left(\dfrac{e+\cos\varphi}{\rho} + \cos\varphi\right)F_S\right] \\ \dfrac{\mathrm{d}i}{\mathrm{d}t} = \dfrac{\eta\sqrt{a}}{\sqrt{\mu}}\dfrac{\cos(\varphi+\omega)}{\rho}F_W \\ \dfrac{\mathrm{d}\Omega}{\mathrm{d}t} = \dfrac{\eta\sqrt{a}}{\sqrt{\mu}}\dfrac{\sin(\varphi+\omega)}{\rho\sin i}F_W \\ \dfrac{\mathrm{d}\omega}{\mathrm{d}t} = \dfrac{\eta\sqrt{a}}{\sqrt{\mu}}\left(-\dfrac{\cos\varphi}{e}F_R + \dfrac{1+\rho}{e\rho}\sin\varphi F_S - \dfrac{\cos i\sin(\varphi+\omega)}{\rho\sin i}F_W\right) \\ \dfrac{\mathrm{d}M}{\mathrm{d}t} = n + \dfrac{\eta^2\sqrt{a}}{e\sqrt{\mu}}\left(\dfrac{\rho\cos\varphi - 2e}{\rho}F_R - \dfrac{1+\rho}{\rho}\sin\varphi F_S\right) \end{cases} \tag{1}$$

where a is the semi-major axis, e is the eccentricity, i is the inclination, ω is the argument of perigee, Ω is the right ascension of the ascending node, φ is the true anomaly, M is the mean anomaly, $\eta = \sqrt{1-e^2}$, $\rho = 1 + e\cos\varphi$, $n = \sqrt{\mu/a^3}$; is the mean motion, μ is the standard gravitational parameter, and $\mathbf{F} = F_R\hat{\mathbf{R}} + F_S\hat{\mathbf{S}} + F_W\hat{\mathbf{W}}$ is the disturbing acceleration expressed in the RSW coordinate frame. The $\hat{\mathbf{R}}$, $\hat{\mathbf{S}}$, and $\hat{\mathbf{W}}$ axes point along the orbital radial, circumferential, and normal directions, respectively.

In this paper, the disturbing acceleration consists of two components: the thrust acceleration and the J_2-perturbed acceleration, i.e.,

$$\begin{cases} F_R = u_R + F_{J_2,R} \\ F_S = u_S + F_{J_2,S} \\ F_W = u_W + F_{J_2,W} \end{cases} \tag{2}$$

where (u_R, u_S, u_W) and $(F_{J_2,R}, F_{J_2,S}, F_{J_2,W})$ denote the components of the thrust acceleration and the J_2-perturbed acceleration in the RSW coordinate frame, respectively. The J_2-perturbed acceleration is given by [5]

$$\begin{cases} F_{J_2,R} = -\dfrac{3\mu J_2 R_\oplus^2}{2r^4}[1 - 3\sin^2 i \sin^2(\omega+\varphi)] \\ F_{J_2,S} = -\dfrac{3\mu J_2 R_\oplus^2}{r^4}\sin^2 i \sin(\omega+\varphi)\cos(\omega+\varphi) \\ F_{J_2,W} = -\dfrac{3\mu J_2 R_\oplus^2}{r^4}\sin i \cos i \sin(\omega+\varphi) \end{cases} \tag{3}$$

where $R_\oplus = 6378.14$ km denotes the Earth's mean radius, $J_2 = 1.082627 \times 10^{-3}$, and r denotes the orbital radius.

For simplicity, the coupling between the J_2 perturbation and the thrust acceleration is ignored, and then the secular variation rate of the orbital element is approximated as

$$\dot{\bar{x}} \approx \dot{\bar{x}}_u + \dot{\bar{x}}_{J_2} \tag{4}$$

where $\dot{\bar{x}}_u$ and $\dot{\bar{x}}_{J_2}$ denote the secular variation rates of the orbital element under the thrust acceleration solely and the J_2 perturbation solely, respectively. The secular variation rates of orbital elements solely under J_2 perturbation are [5]

$$\begin{pmatrix} \dot{\bar{a}}_{J_2} \\ \dot{\bar{e}}_{J_2} \\ \dot{\bar{i}}_{J_2} \\ \dot{\bar{\Omega}}_{J_2} \\ \dot{\bar{\omega}}_{J_2} \\ \dot{\bar{M}}_{J_2} \end{pmatrix} = \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2\cos i \\ 5\cos^2 i - 1 \\ \eta(3\cos^2 i - 1) \end{pmatrix} \tag{5}$$

where $\gamma = J_R / \left(a^{7/2}\eta^4\right)$, and where $J_R = 3J_2 R_\oplus^2 \sqrt{\mu}/4$.

Based on the orbital averaging theory, the secular variation rate of the orbital element solely under thrust acceleration within a time period $[t_0, t_1]$ is

$$\dot{\bar{x}}_u = \frac{1}{\Delta T}\int_{t_0}^{t_1} \dot{x}_u \, \mathrm{d}t \tag{6}$$

where $\Delta T = t_1 - t_0$, and $\dot{x}_u$ denotes the instantaneous orbital variation rate only under thrust acceleration, which is obtained from the GVEs in Eq. (1). Based on the following relationship [6],

$$\mathrm{d}t = \frac{1 - e\cos E}{n}\mathrm{d}E \tag{7}$$

where E is eccentric anomaly, Eq. (6) can be rewritten as

$$\bar{\dot{x}}_u = \frac{1}{n\Delta T}\int_{E_0}^{E_1} (1 - e\cos E)\dot{x}_u \,\mathrm{d}E \tag{8}$$

Substituting Eq. (1) into Eq. (8) yields

$$\bar{\dot{a}}_u = \frac{2\sqrt{a^3}}{n\Delta T\sqrt{\mu}}\int_{E_0}^{E_1} (e\sin E u_R + \eta u_S)\,\mathrm{d}E \tag{9}$$

$$\bar{\dot{e}}_u = \frac{\eta\sqrt{a}}{n\Delta T\sqrt{\mu}}\int_{E_0}^{E_1} \left\{\eta\sin E u_R + \left[2\cos E - \frac{3}{2}e - \frac{1}{2}e\cos(2E)\right]u_S\right\}\mathrm{d}E \tag{10}$$

$$\begin{aligned}\bar{\dot{i}}_u &= \frac{\sqrt{a}}{2n\Delta T\eta\sqrt{\mu}}\int_{E_0}^{E_1} u_W\left[2(1+e^2)\cos\omega\cos E - 3e\cos\omega\right]\,\mathrm{d}E\\ &+ \frac{\sqrt{a}}{2n\Delta T\eta\sqrt{\mu}}\int_{E_0}^{E_1} u_W\left[-2\eta\sin\omega\sin E - e\cos\omega\cos(2E) + e\eta\sin\omega\sin(2E)\right]\,\mathrm{d}E\end{aligned} \tag{11}$$

$$\begin{aligned}\bar{\dot{\Omega}}_u &= \frac{\sqrt{a}}{2n\Delta T\eta\sqrt{\mu}\sin i}\int_{E_0}^{E_1} u_W\left[2\eta\cos\omega\sin E + 2(1+e^2)\sin\omega\cos E\right]\,\mathrm{d}E\\ &+ \frac{\sqrt{a}}{2n\Delta T\eta\sqrt{\mu}\sin i}\int_{E_0}^{E_1} u_W\left[-3e\sin\omega - e\eta\cos\omega\sin(2E) - e\sin\omega\cos(2E)\right]\,\mathrm{d}E\end{aligned} \tag{12}$$

$$\begin{aligned}\bar{\dot{\omega}}_u &= -\bar{\dot{\Omega}}_u\cos i + \frac{\sqrt{a}}{en\Delta T\sqrt{\mu}}\int_{E_0}^{E_1} -\eta(\cos E - e)u_R\,\mathrm{d}E\\ &+ \frac{\sqrt{a}}{en\Delta T\sqrt{\mu}}\int_{E_0}^{E_1}\left[(2-e^2)\sin E - \frac{1}{2}e\sin(2E)\right]u_S\,\mathrm{d}E\end{aligned} \tag{13}$$

$$\begin{aligned}\bar{\dot{M}}_u &= n + \frac{\sqrt{a}}{n\Delta Te\sqrt{\mu}}\int_{E_0}^{E_1}\left[(1+3e^2)\cos E - e^3\cos(2E) - 3e\right]u_R\,\mathrm{d}E\\ &- \frac{\eta\sqrt{a}}{2n\Delta Te\sqrt{\mu}}\int_{E_0}^{E_1}\left[2(2-e^2)\sin E - e\sin(2E)\right]u_S\,\mathrm{d}E\end{aligned} \tag{14}$$

The analytical expressions of the integrals in Eq. (9)-Eq. (14) are derived based on the thrust series approximation in this section.

When the real thrust acceleration within the interval $[E_0, E_1]$ is periodic with respect to eccentric anomaly E, it is approximated by using the truncated Fourier series with respect to E,

$$\begin{cases} u_R \approx \sum_{j=0}^{N_R}\kappa_{R,j}\cos(jLE) + \sum_{j=1}^{N_R}\kappa_{R,j+N_R}\sin(jLE)\\ u_S \approx \sum_{j=0}^{N_S}\kappa_{S,j}\cos(jLE) + \sum_{j=1}^{N_S}\kappa_{S,j+N_S}\sin(jLE)\\ u_W \approx \sum_{j=0}^{N_W}\kappa_{W,j}\cos(jLE) + \sum_{j=1}^{N_W}\kappa_{W,j+N_W}\sin(jLE)\end{cases} \tag{15}$$

where $L = 2\pi/(E_1 - E_0)$, N_χ $(\chi = R, S, W)$ denotes the thrust series truncation order, and $\kappa_{\chi,j}$ $(j = 0, 1, \ldots, 2N_\chi)$ denotes the thrust series coefficient, which is obtained by using the discrete least-squares approximation method [7]. In the case when the real thrust acceleration within the interval

$[E_0, E_1]$ is not periodic with respect to eccentric anomaly, it is extended to the interval $[E_0, E_2]$, where $E_2 = 2E_1 - E_0$, and the thrust acceleration within the interval $[E_1, E_2]$ is an even symmetry of that within the interval $[E_0, E_1]$. Thereby, the truncated Fourier series approximation of the real thrust acceleration within the interval $[E_0, E_1]$ is also obtained with $L = \pi/(E_1 - E_0)$. The obtained thrust Fourier series approximation is then substituted into Eq. (9)-Eq. (14) to give

$$\bar{\dot{a}}_u = \frac{2\sqrt{a^3}}{n\sqrt{\mu}\Delta T}\left(\sum_{j=0}^{2N_R} e m_{3,j}\kappa_{R,j} + \sum_{j=0}^{2N_S} \eta m_{1,j}\kappa_{S,j}\right) \tag{16}$$

$$\bar{\dot{e}}_u = \frac{\eta\sqrt{a}}{n\sqrt{\mu}\Delta T}\left[\sum_{j=0}^{2N_R} \eta m_{3,j}\kappa_{R,j} + \sum_{j=0}^{2N_S}\left(2m_{2,j} - \frac{3}{2}em_{1,j} - \frac{1}{2}em_{4,j}\right)\kappa_{S,j}\right] \tag{17}$$

$$\begin{aligned}\bar{\dot{i}}_u = &\frac{\sqrt{a}}{2n\eta\sqrt{\mu}\Delta T}\sum_{j=0}^{2N_W}\left[2(1+e^2)\cos\omega m_{2,j} - 3e\cos\omega m_{1,j}\right]\kappa_{W,j} \\ &+ \frac{\sqrt{a}}{2n\eta\sqrt{\mu}\Delta T}\sum_{j=0}^{2N_W}\left(-2\eta\sin\omega m_{3,j} - e\cos\omega m_{4,j} + e\eta\sin\omega m_{5,j}\right)\kappa_{W,j}\end{aligned} \tag{18}$$

$$\begin{aligned}\bar{\dot{\Omega}}_u = &\frac{\sqrt{a}}{2n\eta\Delta T\sqrt{\mu}\sin i}\sum_{j=0}^{2N_W}\left[2\eta\cos\omega m_{3,j} + 2(1+e^2)\sin\omega m_{2,j}\right]\kappa_{W,j} \\ &+ \frac{\sqrt{a}}{2n\eta\Delta T\sqrt{\mu}\sin i}\sum_{j=0}^{2N_W}\left(-3e\sin\omega m_{1,j} - e\eta\cos\omega m_{5,j} - e\sin\omega m_{4,j}\right)\kappa_{W,j}\end{aligned} \tag{19}$$

$$\bar{\dot{\omega}}_u = -\bar{\dot{\Omega}}_u\cos i + \frac{\sqrt{a}}{en\Delta T\sqrt{\mu}}\sum_{j=0}^{2N_R}\left[-\eta m_{2,j} + \eta e m_{1,j} + (2-e^2)m_{3,j} - \frac{e}{2}m_{5,j}\right]\kappa_{R,j} \tag{20}$$

$$\begin{aligned}\bar{\dot{M}}_u = &\, n + \frac{\sqrt{a}}{ne\Delta T\sqrt{\mu}}\sum_{j=0}^{2N_R}\left[(1+3e^2)m_{2,j} - e^3 m_{4,j} - 3em_{1,j}\right]\kappa_{R,j} \\ &- \frac{\eta\sqrt{a}}{2ne\Delta T\sqrt{\mu}}\sum_{j=0}^{2N_S}\left[2(2-e^2)m_{3,j} - em_{5,j}\right]\kappa_{S,j}\end{aligned} \tag{21}$$

where

$$\begin{cases} m_{1,j} = \int_{E_0}^{E_1} f(jLE)\,\mathrm{d}E \\ m_{2,j} = \int_{E_0}^{E_1} f(jLE)\cos E\,\mathrm{d}E \\ m_{3,j} = \int_{E_0}^{E_1} f(jLE)\sin E\,\mathrm{d}E \\ m_{4,j} = \int_{E_0}^{E_1} f(jLE)\cos(2E)\,\mathrm{d}E \\ m_{5,j} = \int_{E_0}^{E_1} f(jLE)\sin(2E)\,\mathrm{d}E \end{cases} \tag{22}$$

and where

$$f(jLE) = \begin{cases} \cos(jLE), & \text{if } 0 \le j \le N_\chi \\ \sin[(j - N_\chi)LE], & \text{if } N_\chi + 1 \le j \le 2N_\chi \end{cases} \tag{23}$$

The integrals in Eq. (22) can be obtained from the following relationships:

$$\int_{E_0}^{E_1} \cos(b_1 E)\cos(b_2 E)\,\mathrm{d}E = \begin{cases} \left.\dfrac{E}{2} + \dfrac{\sin[(b_1+b_2)E]}{2(b_1+b_2)}\right|_{E_0}^{E_1}, & \text{if } b_1 = b_2 \\ \left.\dfrac{\sin[(b_1-b_2)E]}{2(b_1-b_2)} + \dfrac{\sin[(b_1+b_2)E]}{2(b_1+b_2)}\right|_{E_0}^{E_1}, & \text{if } b_1 \ne b_2 \end{cases} \tag{24}$$

$$\int_{E_0}^{E_1} \cos(b_1 E)\sin(b_2 E)\,\mathrm{d}E = \begin{cases} \left.-\dfrac{\cos[(b_2+b_1)E]}{2(b_2+b_1)}\right|_{E_0}^{E_1}, & \text{if } b_1 = b_2 \\ \left.-\dfrac{\cos[(b_2+b_1)E]}{2(b_2+b_1)} - \dfrac{\cos[(b_2-b_1)E]}{2(b_2-b_1)}\right|_{E_0}^{E_1}, & \text{if } b_1 \ne b_2 \end{cases} \tag{25}$$

$$\int_{E_0}^{E_1} \sin(b_1 E)\sin(b_2 E)\,\mathrm{d}E = \begin{cases} \left.\dfrac{E}{2} - \dfrac{\sin[(b_1+b_2)E]}{2(b_1+b_2)}\right|_{E_0}^{E_1}, & \text{if } b_1 = b_2 \\ \left.\dfrac{\sin[(b_1-b_2)E]}{2(b_1-b_2)} - \dfrac{\sin[(b_1+b_2)E]}{2(b_1+b_2)}\right|_{E_0}^{E_1}, & \text{if } b_1 \ne b_2 \end{cases} \tag{26}$$

Based on Eq. (4), Eq. (5), and Eq. (16)-Eq. (21), one can obtain the analytical approximation expressions of the secular variation rates of orbital elements under thrust acceleration and J_2 perturbation. The analytical propagation of the secular orbital motion can be obtained as

$$\begin{cases} a_s(t) = a(t_0) + \bar{\dot{a}} \cdot (t - t_0) \\ e_s(t) = e(t_0) + \bar{\dot{e}} \cdot (t - t_0) \\ i_s(t) = i(t_0) + \bar{\dot{i}} \cdot (t - t_0) \\ \Omega_s(t) = \Omega(t_0) + \bar{\dot{\Omega}} \cdot (t - t_0) \\ \omega_s(t) = \omega(t_0) + \bar{\dot{\omega}} \cdot (t - t_0) \\ M_s(t) = M(t_0) + \bar{\dot{M}} \cdot (t - t_0) \end{cases} \tag{27}$$

The above equation can be rewritten in the state-space form,

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{x}_0 + (\bar{\dot{\boldsymbol{x}}}_{J_2} + \bar{\dot{\boldsymbol{x}}}_u) \cdot (t - t_0) \\ &= \boldsymbol{x}_0 + \bar{\dot{\boldsymbol{x}}}_{J_2} \cdot (t - t_0) + \boldsymbol{QK} \cdot (t - t_0) \end{aligned} \tag{28}$$

where $\boldsymbol{x} = [a, e, i, \Omega, \omega, M]^{\mathrm{T}}$ denotes the orbital element vector, and the thrust coefficient vector is

$$\boldsymbol{K} = [\kappa_{R,0}, \ldots, \kappa_{R,2N_R}, \kappa_{S,0}, \ldots, \kappa_{S,2N_S}, \kappa_{W,0}, \ldots, \kappa_{W,2W_R},]^{\mathrm{T}} \tag{29}$$

and the matrix $\boldsymbol{Q} \in \mathbf{R}^{6\times(2N_R+2N_S+2N_W+3)}$ can be obtained from Eq. (16)-Eq. (21).

III. Control-distance metric distribution

The control-distance metric is defined as [1]

$$J = \int_{t_0}^{t_f} \boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{u}(t)\,\mathrm{d}t \tag{30}$$

Substituting the thrust series expansion Eq. (15) into the above equation yields

$$J = \boldsymbol{K}^{\mathrm{T}}\boldsymbol{G}(t_f, t_0)\boldsymbol{K} \tag{31}$$

where $\boldsymbol{G}$ can be easily derived from the integrations of trigonometric functions.

For the given initial and terminal states, the terminal state constraint is derived based on Eq. (28),

$$\boldsymbol{x}_f = \boldsymbol{x}_0 + \Delta T\bar{\dot{\boldsymbol{x}}}_{J_2} + \Delta T\boldsymbol{Q}\boldsymbol{K} \tag{32}$$

where $\Delta T = t_f - t_0$.

Based on the Lagrange multiplier method, the augmented performance index can be defined as

$$\tilde{J} = \boldsymbol{K}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{K} + \boldsymbol{\lambda}^{\mathrm{T}}(\Delta T\boldsymbol{Q}\boldsymbol{K} + \boldsymbol{x}_0 + \Delta T\bar{\dot{\boldsymbol{x}}}_{J_2} - \boldsymbol{x}_f) \tag{33}$$

The optimal necessary condition is

$$\frac{\partial \tilde{J}}{\partial \boldsymbol{K}} = 2\boldsymbol{G}\boldsymbol{K} + \Delta T\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\lambda} = 0 \tag{34}$$

One can further obtain that

$$\boldsymbol{K} = -\frac{\Delta T}{2}\boldsymbol{G}^{-1}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\lambda} \tag{35}$$

Substituting Eq. (35) into the terminal state constraint Eq. (32) yields

$$\boldsymbol{\lambda} = \frac{2}{\Delta T^2}\left[\boldsymbol{Q}\boldsymbol{G}^{-1}\boldsymbol{Q}^{\mathrm{T}}\right]^{-1}\Delta\boldsymbol{x} \tag{36}$$

where $\Delta \boldsymbol{x} = \boldsymbol{x}_0 + \Delta T \bar{\dot{\boldsymbol{x}}}_{J_2} - \boldsymbol{x}_f$. Furthermore, taking Eq. (36) into Eq. (35) yields

$$\boldsymbol{K} = -\frac{1}{\Delta T}\boldsymbol{G}^{-1}\boldsymbol{Q}^{\mathrm{T}}\left[\boldsymbol{Q}\boldsymbol{G}^{-1}\boldsymbol{Q}^{\mathrm{T}}\right]^{-1}\Delta \boldsymbol{x} \tag{37}$$

Then, the analytical solution to the control-distance metric is

$$\boxed{J(\boldsymbol{x}_0, \boldsymbol{x}_f) = \frac{1}{\Delta T^2}\Delta \boldsymbol{x}^{\mathrm{T}}\left[\boldsymbol{Q}\left(\frac{\boldsymbol{x}_0 + \boldsymbol{x}_f}{2}\right)\boldsymbol{G}^{-1}\boldsymbol{Q}^{\mathrm{T}}\left(\frac{\boldsymbol{x}_0 + \boldsymbol{x}_f}{2}\right)\right]^{-1}\Delta \boldsymbol{x}} \tag{38}$$

Based on this expression, the probability distribution of the control-distance metric can be obtained for the uncertain initial and terminal states. If the analytical expression of the term $\left[\boldsymbol{Q}\boldsymbol{G}^{-1}\boldsymbol{Q}^{\mathrm{T}}\right]^{-1}$ can be obtained, the second-order partial derivatives of the control-distance metric with respect to the initial and terminal states can be analytically derived; otherwise, the differential algebra (DA) tool is used to compute the high-order partial derivatives numerically. On this basis, the GMM method can be further incorporated to tackle the non-Gaussian distribution problem.

References

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